Uncertain Data Management
Extensional And Intensional Query Processing

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Independent Queries

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- take two relational atoms L_1 and L_2
- they *unify* if we find substitutions under which the two atoms become the same, i.e., they have a common image
- two queries Q_1 and Q_2 are independent if no two atoms unify

Independent Queries

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- no common image possible (why?)

Independent Queries

Proposition

If $Q_1, Q_2, \ldots, Q_k$ are syntactically independent queries, then $Q_1, Q_2, \ldots, Q_k$ are independent probabilistic events.

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If $Q_1, Q_2, \ldots, Q_k$ *are syntactically independent queries, then* $Q_1, Q_2, \ldots, Q_k$ *are independent probabilistic events.*

- extensional query evaluation: apply simple probabilistic rules on independent parts of queries

Extensional Rules

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Negation $\mathrm{P}(\neg Q) = 1 - \mathrm{P}(Q)$

Extensional Rules

For a query Q of the form $Q = \exists x.Q'$:

- x is a root variable if every atom $L \in Q'$ contains the variable x
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- for any two constants $a \neq b$, $Q'(a/x)$ and $Q'(b/x)$ are syntactically independent; and
- **Independent Project** rule:

$$\mathrm{P} = 1 - \prod_{a \in \mathsf{ADom}} \left(1 - \mathrm{P}\left(Q'\left(a/x\right)\right)\right)$$

Extensional Rules

Inclusion-Exclusion For a query $Q = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k$ ($Q_1, \ldots, Q_k$ not necessarily independent):

$$\mathrm{P}(Q) = -\sum_{s \subseteq [k], s \neq \emptyset} (-1)^{|s|} \mathrm{P}(\bigvee_{i \in [s]} Q_i)$$

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Applying Extensional Rules: Algorithm

- start from Q and write $\mathrm{P}(Q)$ in terms of $\mathrm{P}(Q_1), \mathrm{P}(Q_2), \ldots$

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- if the algorithm stops at ground tuples, then query is safe, otherwise unsafe
- algorithm is non-deterministic, e.g., inclusion-exclusion
- if we can evaluate $\mathrm{P}(Q)$ using rules except inclusion-exclusion (exponential rewriting), then $\mathrm{P}(Q)$ is tractable

Simple Example

$$Q = R(x), S(x, y) = \exists x.(R(x) \wedge \exists y.S(x, y))$$

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$$\begin{aligned}
\mathrm{P}(Q) &= 1 - \prod_{a \in \mathsf{ADom}} \left(1 - \mathrm{P}\left(R(a) \wedge \exists y.S(a,y)\right)\right) \\
&= 1 - \prod_{a \in \mathsf{ADom}} \left(1 - \mathrm{P}\left(R(a)\right) \cdot \mathrm{P}\left(\exists y.S(a,y)\right)\right)
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&= 1 - \prod_{a \in \mathsf{ADom}} \left(1 - \mathrm{P}\left(R(a)\right) \cdot \prod_{b \in \mathsf{ADom}} \left(1 - \mathrm{P}\left(S(a,b)\right)\right)\right)
\end{aligned}$$

Exercise: Extensional Query

Booking

date	**teacher**	**room**
3011	C42	p_1
0712	C42	p_2
1412	C017	p_3
0401	C017	p_4

Room

room	**equipment**	
C42	projector	q_1
C42	none	$1 - q_1$
C017	projector	q_2
C017	none	$1 - q_2$

- calculus: $Q() : \; \exists d, r. B(d, r) \wedge R(r, \text{'none'})$

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We want to extend relational database plans to probabilistic databases:

- any safe query has a safe plan (=plan which computes probabilities correctly)
- can use it in "normal" DBMS to compute output probabilities (benefiting from optimization, parallelism, ...)
- still can compute if queries are unsafe, but probabilities are incorrect (may be able to compute upper and lower bounds)

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Let us see how the operators of relational algebra $(\bowtie, \sigma, \pi, \cup)$ are implemented

Extensional Operators

Independent Join $\bowtie$

$$R \bowtie^{i}_{C} S = \{(a, b, p_R(a) \cdot p_S(b)) \mid a \in \Pi_A(R), b \in \Pi_B(S), (a, b) \in \Pi_A(R) \bowtie_C \Pi_B(S)\}$$

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Independent Project π – $u_1, \ldots, u_k$ are the attributes that have a common value a

$$\pi_a^i(R) = \left\{ \left(a, 1 - \prod_{u \in R : u.A = a} (1 - u.p) \right) \mid a \in \Pi_A(R) \right\}$$

Extensional Operators

Independent Union $\cup$ – for two relations $R(A_1, p)$, $S(A_2, p)$

$$R \cup^i_A S = \left\{ (a, 1 - (1 - p_R(a.A_1))\,(1 - p_S(a.A_2))) \mid a.A_1 \in \Pi_{A_1}(R) \vee a.A_2 \in \Pi_{A_2}(S) \right\}$$

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$$\sigma_C(R) = \{(a, p_R(a)) \mid C \models a\}$$

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Complementation

$$C_A(R) = \left\{ (a, 1 - p_R(a)) \mid a \in \mathsf{ADom}(D)^k \right\}$$

Extensional Plan Example

Booking

teacher	room	
Antoine	C42	p_1
Antoine	C42	p_2
Silviu	C017	p_3
Silviu	C018	p_4

Room

room	equipment	
C42	projector	q_1
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Who are the teachers teaching in rooms with projectors ?

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- $\pi_{\text{teacher}}\left(B \bowtie \pi_{\text{room}}(\sigma_{\text{'projector'}}(R))\right)$

Extensional Plan Example

B

teacher	room	
Antoine	C42	p_1
Antoine	C42	p_2
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$\pi_{\text{room}}(\sigma_{\text{'projector'}}(R))$

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Extensional Plan Example

$B \bowtie \pi_{\text{room}}(\sigma_{\text{'projector'}}(R))$

teacher	room	
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Extensional Plan Example

$\pi_{\text{teacher}}\left(B \bowtie \pi_{\text{room}}(\sigma_{\text{'projector'}}(R))\right)$

teacher	
Antoine	$1-(1-p_1q_1)(1-p_2q_1)$
Silviu	$1-(1-p_3q_2)(1-p_4q_3)$

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Plans For Unsafe Queries

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However, we can compute upper bounds in some cases:

Proposition

Let $Q = Q_1 \vee Q_2 \vee \cdots \vee Q_k$ and Q_i a conjunctive query without self-joins (no joins between the same relation names). Then any plan using independent join, indepedent projection, independent union and selection will compute an upper bound for the answer tuple probabilities.

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Query (unsafe) $Q : R(z, x), S(x, y), T(y)$

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P_1 =& \pi_z(\pi_{zx}(R(z, x) \bowtie S(x, y)) \bowtie T(y)) \\
P_2 =& \pi_z(R(z, x) \bowtie \pi_x(S(x, y) \bowtie T(y))) \\
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The above Proposition gives us a way to compute tighter upper bounds $\rightsquigarrow$ execute as many plans as possible an take the minimum as the estimated probability

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Ingredient – same concept of independence between two lineages:

- Q_1 and Q_2 are independent if they have *disjoint supports* $(\mathsf{Var}(Q_1) \cap \mathsf{Var}(Q_2) \neq \emptyset)$

Intensional Query Rules

Independent AND: $\mathrm{P}(\Phi_{Q_1 \wedge Q_2}) = \mathrm{P}(\Phi_{Q_1}) \cdot \mathrm{P}(\Phi_{Q_2})$

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Negation: $\mathrm{P}(\neg\Phi_Q) = 1 - \mathrm{P}(\Phi_Q)$

Intensional Query Rules

Disjoint OR – two formulas Φ_1, Φ_2 are called disjoint if the formula $\Phi_1 \wedge \Phi_2$ is not satisfiable $\mathrm{P}(\Phi_1 \vee \Phi_2) = \mathrm{P}(\Phi_1) + \mathrm{P}(\Phi_2)$

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Shannon expansion – a general rule; intuitively, choose a variable to instantiate and rewrite Φ

$$\mathrm{P}(\Phi) = \sum_{i=0,m} P(\Phi|_{x=a_i}) \cdot P(X = a_i)$$

Intensional Query Evaluation Using Rules

Algorithm:

- same as query evaluation in the extensional case: iteratively apply the rules until we arrive at ground tuples/variable;
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Complexity:

- the algorithm is non-deterministic
- if only independent OR, AND, negation are applied, then the size of the probability formula is linear in Φ – size lower bound
- if only Shannon expansion can be used the formula is exponential in the size of Φ – size upper bound

Exercise: Lineage

$$Q() \iff x_1\neg y_1 \vee x_2\neg y_1 \vee x_3\neg y_2 \vee x_4\neg y_2$$

(assume $\mathrm{P}(x_i) = p_i$ and $\mathrm{P}(y_i) = q_i$)

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Read-Once Formulas

$$\Phi = x_1\neg y_1 \vee x_2\neg y_1 \vee x_3\neg y_2 \vee x_4\neg y_2$$
$$\Phi' = \neg y_1(x_1 \vee x_2) \vee \neg y_2(x_3 \vee x_4)$$

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Definition

A formula ϕ is *read-once* iff there exists Φ' such that no variable is repeated more than once in it.

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Efficient class of formulas:

- if read-once Φ' of Φ exists, it can be computed from Φ in polynomial times
- $\mathrm{P}(\Phi')$, where Φ' read-once, can be computed in linear time by applying only independent AND, OR and negation

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A circuit for Φ is a rooted, labeled DAG, containing a subset of the following types of gates:

1. *Independent AND*: labeled $\wedge$ having children variables or clauses,
2. *Independent/Disjoint OR* labeled $\vee$ having as children variables or clauses,
3. *NOT* labeled $\neg$, child the negated clause/variable
4. *Conditional gate* labeled with a variable X_i and having two edges corresponding to the clauses occurring when $X_i =$ true and $X_i =$ false (Shannon expansion),
5. *Leaf node*, either 1 or 0, or the variable X_i.

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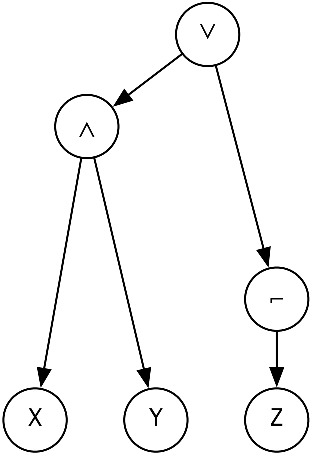
Several possible compilation targets: read-once formulas, d-DNNF$^\neg$, OBDD, FBDD

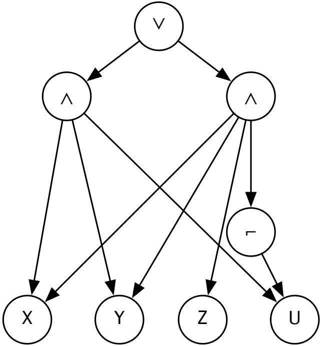
Read-Once Circuits



A read-once circuit contains only independent AND, independent OR, and NOT gates, and variables on leaf nodes.

Given a read-once circuit representing a formula Φ, one can compute $\mathrm{P}(\Phi)$in linear time. How?

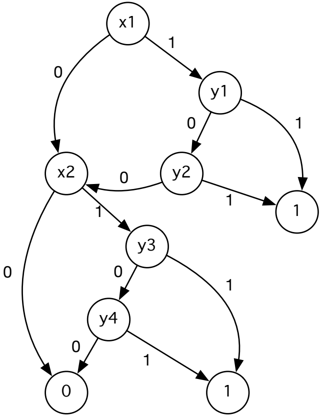
d-DNNF$^\neg$ (Deterministic Decomposable Negation Normal Form)



A d-DNNF$^\neg$ contains only independent AND, disjoint OR, and NOT gates, and variables on leaf nodes.

Given a d-DNNF$^\neg$ representing a formula Φ, one can compute $\mathrm{P}(\Phi)$in linear time. How?

FBDD/OBDD (Free/Ordered Binary Decision Diagram)



A FBDD contains only conditional gates as nodes, and the leafs are either 1 or 0.

An OBDD is an FBDD with the property that all paths from the root to the leaves visit the nodes in the same order.

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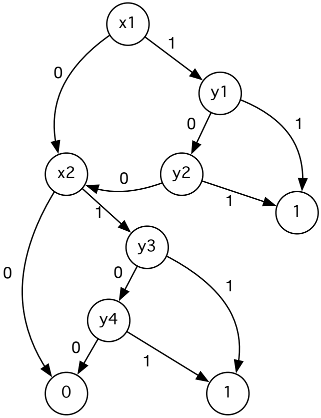

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- a read-once expression Φ has an OBDD of linear size
- OBDDs can be build inductively on the structure of a formula Φ: $\Phi_1 \wedge \Phi_2$ or $\Phi_1 \vee \Phi_2$ have width[a] at most $w_1 w_2$

[a]width – the highest number of conditional gates

Exercise: Lineage Circuits

Compile $\Phi = x_1 y_1 \vee x_2 \neg y_1 \vee \neg x_2 \neg y_2 \vee \neg x_1 y_2$ into a d-DNNF$^\neg$.

Compile $\Phi = x_1 \neg y_1 \vee x_2 \neg y_1 \vee x_3 \neg y_2 \vee x_4 y_2$ into an OBDD and a read-once circuit.

Circuit Representations: Can We Always Use Them?

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Not all formulas can be represented efficiently with circuits. *What if we are happy with an estimated probability?*

Approximating Φ: Upper, Lower Bounds

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Iteratively use formulas for $\wedge$ and $\vee$:

$$
\begin{aligned}
\max(\mathrm{P}(\Phi_1), \mathrm{P}(\Phi_2)) \leqslant \quad & \mathrm{P}(\Phi_1 \vee \Phi_2) \leqslant & \min(\mathrm{P}(\Phi_1) + \mathrm{P}(\Phi_2), 1) \\
\max(0, \mathrm{P}(\Phi_1) + \mathrm{P}(\Phi_2) - 1) \leqslant \quad & \mathrm{P}(\Phi_1 \wedge \Phi_2) \leqslant & \min(\mathrm{P}(\Phi_1), \mathrm{P}(\Phi_2)) \\
& \mathrm{P}(\neg\Phi) = & 1 - \mathrm{P}(\Phi)
\end{aligned}
$$

Approximating Φ: Monte-Carlo

We can use Monte-Carlo algorithms (sampling) for estimating any expression Φ , by repeating a sampling process N times:

1. choose a random valuation $\theta \in w(\Phi)$ proportionally to the probability $\mathrm{P}(\theta) \rightsquigarrow$ each variable X in Φ is set to 1 with probability $\mathrm{P}(X)$
2. if $\Phi(\theta)$ is true, then return $Z = 1$, otherwise $Z = 0$

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How many samples do we need? Use Chernoff bounds to get an estimation

$$N = \lceil \frac{4 \log \frac{2}{\delta}}{p\epsilon^2} \rceil$$

Approximating Φ

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More advanced estimators can be used if we assume the lineage is represented in a certain way.

Example: if our formula is in **DNF** (Disjunctive Normal Form), then one can use the Karp-Luby estimator to efficiently compute probabilities.